low shell of square planform is shown in Fig. 4. The variation of the lowest frequency with the curvature ratio of shallow shell with square planform is shown in Fig. 5. The depth ratio and width ratio of the stiffeners are the same as the previous examples.

Conclusions

The free vibrations of a stiffened shallow shell are studied numerically by use of the collocation method on the basis of the theory of thin orthotropic shallow shells. A comparison of the present results with earlier results[1,2] shows good agreement in the case of the stiffened plate.

For a unidirectionally stiffened shallow shell, the lowest frequency increases considerably with the depth ratio of the stiffener. The influence of the width ratio on the lowest frequency is negligible expectedly. The number of the stiffeners affects the dynamic behavior of the stiffened shallow shell only for certain aspect ratios. The lowest frequency increases considerably for certain range of the orthotropicity ratio. When increasing the curvature thickness ratio, the lowest frequency decreases sharply and approaches to a limit value.

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Modal Sensitivities for Repeated Eigenvalues and Eigenvalue Derivatives

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I. Introduction

RESEARCH on the modal sensitivites, i.e., the determination of derivatives of eigenvalues and eigenvectors with respect to system parameters, has been studied for quite some time, motivated mainly by its important applications in areas such as system design, system identification, and optimization. An excellent survey paper by Adelman and Haftka[1] summarizes the progress made in the study of modal sensitivities.

Received Aug. 10, 1990; revision received Dec. 1, 1990; accepted for publication Dec. 7, 1990. Copyright © 1991 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved.

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The first computationally efficient algorithm for computing the derivatives of eigenvalues and eigenvectors of real eigensystems with nonrepeated eigenvalues was introduced by Nelson[2] in 1976. It preserves the symmetry and bandedness of the original eigensystem and requires only those eigendata that are to be differentiated. Many real systems, though, exhibit repeated eigenvalues or identical frequencies and have different mode shapes. Nelson's algorithm cannot be directly applied to such systems. Ojalvo[3] extended Nelson's algorithm for solving such singularity problems arising in real symmetric systems, which was later completed independently by Mills-Curran[4] and Dailey.[5] The extended Nelson's algorithm, however, is valid only when the eigenvalue derivatives are assumed to be distinct. In fact, it will be shown that Dailey's argument for the case when both repeated eigenvalues and repeated eigenvalue derivatives are present is not correct.

This Note extends the method of Refs. 3–5 for computing the derivatives of eigenvalues and eigenvectors to include the case when both repeated eigenvalues and repeated eigenvalue derivatives are present. A symmetrical system subjected to parameter perturbations (e.g., a 4×4 cyclic matrix representing a rotor of four blades with the natural frequencies of two of its adjacent blades subjected to a small perturbation) is such a case. The present extension is based on utilizing the third derivative of the eigenvalue problem.

II. Theoretical Background

Consider the eigenvalue problem

$$(K-\lambda M)x = 0 \tag{1}$$

$$x^T M x = 1 \tag{2}$$

where M and K are $n\times n$ real symmetric matrices whose elements depend continuously on system parameters, λ is an eigenvalue, and x is the corresponding eigenvector. Moreover, it is assumed that the eigenvalue problem has m repeated eigenvalues, λ_i, $i=1,2,\ldots,m$ when a system parameter $p=p_0$.

The objective is to determine the derivatives of the repeated eigenvalues and the derivatives of the corresponding eigenvectors, at $p=p_0$. Let $\lambda_1=\lambda_2=\ldots=\lambda_m=\lambda$ denote the m repeated eigenvalues, and $X\in R^{n\times m}$ be the m arbitrary eigenvectors associated with λ at $p=p_0$. The eigenvalue problem corresponding to the case of repeated eigenvalues can be written as

$$KX = MX\Lambda \tag{3}$$

$$X^T MX = I \tag{4}$$

with $\Lambda=\lambda I\in R^{m\times m}$. Consequently, the derivatives of the eigenvalues and the eigenvectors with respect to the system parameter p, at p_0, may be obtained by taking the derivatives of Eqs. (3) and (4) with respect to p. However, the derivatives of the eigenvectors may not exist since it is possible that X may not be continuous and differentiable in p because of the nonuniqueness of X. Therefore, a correct set of eigenvectors (denoted by Z), which are continuous and differentiable in p associated with λ, is needed in Eqs. (3) and (4) in order for the eigenvector derivatives to exist.

Because Z and X are both matrices of eigenvectors, there exists a transformation A such that $Z=XA$ where A is orthogonal (i.e., $A^TA=I$) due to the orthonormal constraint of the eigenvectors. References 4 and 5 give an algorithm, which is briefly described below, for computing A, Λ', and Z' where ($'$) denotes the derivative with respect to p.

Taking the derivative of the eigenvalue equation $KZ=MZ\Lambda$ yields

$$(K-\lambda M)Z' = (\lambda M' - K')Z + MZ\Lambda' \tag{5}$$

Premultiplying by X^T and substituting $Z = XA$ and $X^T(K - \lambda M) = 0$ yields

$$[X^T(K' - \lambda M')X]A \equiv DA = A\Lambda' \tag{6}$$

Thus, a reduced-order eigenvalue problem of order m described by Eq. (6) needs to be solved to determine Λ' and A. In particular, λ_i' and the ith column of A are the ith eigenvalue and the ith eigenvector, respectively, of the symmetric matrix D. Note that if these m λ_i' are distinct, then a unique A is generated, which, in turn, gives the unique and correct Z needed to compute the unique eigenvector derivatives corresponding to the repeated eigenvalues. However, the case when some eigenvalue derivatives are repeated, even when A exists, is complicated and is not correctly handled by Dailey.[5] We study the latter case in Sec. III.

To solve for Z', one now returns to Eq. (5). However, from Eq. (5), one cannot uniquely determine Z' since $(K - \lambda M)$ has rank $n - m$ and a kernel spanned by the columns of Z. Consequently, if V is a solution to Eq. (5), so is $V + ZC$ where C is an $m \times m$ arbitrary constant matrix. Hence, the eigenvector derivatives are assumed to have the form[2-5]

$$Z' = V + ZC \tag{7}$$

where V satisfies the modified Eq. (5):

$$\bar{G}V = \bar{F} \tag{8}$$

where $G = (K - \lambda M)$, $F = (\lambda M' - K')Z + MZ\Lambda'$. Algorithms for zeroing out m rows and columns of G and F to obtain $\bar{G}$ and $\bar{F}$ for uniquely solving for V are given in Refs. 3-5 and are omitted here.

C can be determined in two stages. First, differentiate the orthonormal constraint $Z^TMZ = I$ once and substitute Eq. (7) into the result, giving the diagonal elements of C. Second, the off-diagonal elements can be obtained by differentiating Eq. (5), premultiplying by Z^T, and using Eq. (7) and $\Lambda' = Z^T(K' - \lambda M')Z$. This yields Eqs. (9) and (10) to determine matrix C:

$$C + C^T = -V^TMZ - Z^TMV - Z^TM'Z \equiv Q \tag{9}$$

$$C\Lambda' - \Lambda'C + 0.5\Lambda'' = Z^T(K' - \lambda M')V$$
$$- Z^T(M'Z + MV)\Lambda' + 0.5Z^T(K'' - \lambda M'')Z \equiv R \tag{10}$$

Now, from Eq. (9), it is clear that $c_{ii} = 0.5q_{ii}$ and $c_{ij} + c_{ji} = q_{ij} = q_{ji}$, whereas c_{ij} for $i \neq j$ is indeterminate. The off-diagonal elements c_{ij} are determined from Eq. (10), which indicates a separation of diagonal and off-diagonal elements. In particular, $C\Lambda' - \Lambda'C$ has zero diagonal elements and Λ'' has zero off-diagonal elements. Therefore, the off-diagonal elements c_{ij} and the second derivatives λ_i'' can be determined simultaneously from Eq. (10) as $\lambda_i'' = 2r_{ii}$ and

$$c_{ij} = r_{ij}/(\lambda_j' - \lambda_i') \qquad \text{if} \qquad j \neq i \tag{11}$$

Consequently, Z' is uniquely determined if no repeated eigenvalue derivatives occur. However, when $\lambda_j' = \lambda_i'$, c_{ij} and c_{ji} cannot be determined from Eq. (11). Dailey's argument for this case of repeated eigenvalue derivatives is the following. When repeated eigenvalue derivatives occur, it means that the original m-dimensional eigenspace corresponding to the m repeated eigenvalues is not split into m independent one-dimensional eigenvectors when p is varied from p_0; instead, some of its dimensions stick together. In such cases, the eigenvector derivatives are not unique, thus providing the freedom to arbitrarily set some parameters as long as Eq. (9) is satisfied. Dailey recommends setting $c_{ij} = c_{ji} = 0.5q_{ij}$ whenever $\lambda_i' = \lambda_j'$. The derivative Z' resulting from that will be valid but not unique.

Dailey's recommended solution for repeated eigenvalue derivatives is good only if the eigenvectors associated with the repeated eigenvalue derivatives stick together. This means that some solution curves of eigenvalues as functions of p are inseparable in the neighborhood of p_0 and is the special case where many solution curves simultaneously intersect at a particular point. Moreover, $\lambda_i = \lambda_j$, together with $\lambda_i' = \lambda_j'$ do not necessarily mean that the eigenvalue curves i and j are coalesced in the neighborhood of p_0. The latter becomes obvious by looking at the Taylor series expansions of the two curves at point p_0. Also, because Eq. (11) [deduced from Eq. (10)] is correctly derived, it should always be satisfied as well. When $\lambda_i' = \lambda_j'$, however, Eq. (11) is violated unless $r_{ij} = r_{ji} = 0$, which is not always true due to the nonuniqueness of A (and, hence, Z). Therefore, Dailey's recommended algorithm will not give the correct answer when the eigenvalue derivatives are repeated unless these eigenvalues are indeed inseparable in the neighborhood of p_0. In the following section, we present a new algorithm for such a case.

III. New Algorithm

Let $A = [a_1, a_2, \ldots, a_m]$ where a_i is the ith column of A. Suppose matrix D has $\lambda_1' = \lambda_2' = \ldots = \lambda_k' = \lambda'$ as one group of repeated eigenvalues of multiplicity k. Then a_i, $i = 1, \ldots, k$ are not unique and the correct set of eigenvectors need to be determined. To this end, we introduce another orthogonal transformation $B \in R^{k \times k}$ such that $Z_1 = X[a_1, a_2, \ldots, a_k]B \equiv XA_1B$ are the k correct eigenvectors associated with λ and λ'. Hence, Z can be written as $[Z_1, Z_2]$ where $Z_2 = X[a_{k+1}, \ldots, a_m] \equiv XA_2$ are the eigenvectors associated with the eigenvalue derivatives different from λ'. Also note that the orthonormal constraint on the eigenvectors lead to $B^TB = I$. B may be determined by taking the second derivative of the sub-eigenproblem $KZ_1 - MZ_1\Lambda_1 = 0$ where $\Lambda_1 = \lambda I \in R^{k \times k}$. Differentiating the sub-eigenproblem twice, premultiplying the result by Z_1^T, and using $Z_1^T(K - \lambda M) = 0$ and $Z_1^TMZ_1 = I$, we get

$$Z_1^T(K'' - \lambda M'' - 2\lambda' M')Z_1$$
$$+ 2Z_1^T(K' - \lambda M' - \lambda' M)Z_1' = \Lambda_1'' \tag{12}$$

From Eq. (7), $Z_1' = V_1 + Z_1C_1 + Z_2C_2$, where C_1 and C_2 are submatrices of C. V_1 can be determined by Eq. (8), yielding $V_1 = \bar{G}^{-1}\bar{F}_1Z_1$, where $F_1 \equiv (\lambda M' - K' + \lambda' M)$. Now substituting for Z_1' and $Z_1 = XA_1B$ in Eq. (12), premultiplying by B, and using $BB^T = I$, one obtains

$$[A_1^TX^T(K'' - \lambda M'' - 2\lambda' M' + 2(K' - \lambda M'$$
$$- \lambda' M)\bar{G}^{-1}\bar{F}_1)XA_1]B \equiv D_1B = B\Lambda_1'' \tag{13}$$

where $Z_1^T(K' - \lambda M' - \lambda' M)(Z_1C_1 + Z_2C_2) = 0$ has been used, which is a direct consequence of Eqs. (5) and (6). Note that Eq. (13) is now an eigenvalue equation for matrix $D_1 \in R^{k \times k}$, from which λ_i'', $i = 1, \ldots, k$ and B are obtained. It is clear that unique columns for B can be determined when λ_i'' are distinct. This indicates that the k eigenvalues are split into k distinct values as p varies, even though they have the same slopes (first derivatives) at p_0. Correspondingly, a set of unique eigenvectors Z_1 are obtained. If λ_i'' is itself repeated, then corresponding columns of B are not uniquely determined and another transformation matrix is needed to determine the correct eigenvectors, which can be computed from the information of the third derivatives of the eigenvalue problem. Such cases are not addressed here.

Other groups of repeated eigenvalue derivatives of Eq. (6), if any, can be treated in the same manner; i.e., Eq. (13) is formed to determine an orthogonal transformation for each group. Once the transformation matrices B are obtained, A can be uniquely formed and, subsequently, a unique Z is obtained. Equations (7-10) are then solved for the derivatives Z' and λ_i''. Since partial information from the second derivative equation of the eigensystem has been used to determine unique eigenvectors associated with the repeated eigenvalue derivatives, the eigenvector derivatives with repeated eigenvalue derivatives are not uniquely determined. This is seen easily from Eq. (11), where c_{ij} and c_{ji} are indeterminate when $\lambda_i' = \lambda_j'$ for $j \neq i$, since r_{ij} and r_{ji} are shown to be equal to zero. Eigenvector derivatives with distinct eigenvalue derivatives, however, are uniquely determined.

Information from the third derivative of the eigenproblem is necessary to complete the computation of eigenvector derivatives with repeated eigenvalue derivatives. Differentiating Eq. (5) gives the second derivative of the eigensystem

$$(K - \lambda M)Z'' \equiv GZ'' = 2M'Z\Lambda' + 2MZ'\Lambda' + MZ\Lambda'' + (\lambda M'' - K'')Z + 2(\lambda M' - K')Z' \equiv H \tag{14}$$

The second derivatives of eigenvectors are assumed to have the form $Z'' = U + ZE$ with U computed from the modified Eq. (14) $\bar{G}U = \bar{H}$, which results in a unique solution for U. Again, the procedures in Refs. 3–5 can be used to obtain $\bar{G}$ and $\bar{H}$. Differentiating Eq. (14) again, premultiplying the result by Z^T, and using $Z'' = U + ZE$ and $Z^T(K' - \lambda M')Z = \Lambda'$ yields

$$E\Lambda' - \Lambda' E + \Lambda'''/3 = Z^T(K' - \lambda M')U - Z^T(M''Z + 2M'Z' + MU)\Lambda' + Z^T(K''' - \lambda M''')Z/3 + Z^T(K'' - \lambda M'')Z' - Z^T(M'Z + MZ')\Lambda'' \equiv S \tag{15}$$

which is similar in form to Eq. (10). Separation of the diagonal and off-diagonal elements of the left side of Eq. (15) yields $\lambda_i''' = 3s_{ii}$ and

$$e_{ij}(\lambda_j' - \lambda_i') = s_{ij} \qquad \text{if} \qquad j \neq i \tag{16}$$

From Eq. (16), s_{ij} and s_{ji} must vanish when $\lambda_j' = \lambda_i'$. Hence, $s_{ji} = s_{ij} = 0$ from which the indeterminate c_{ij} and c_{ji} of Eq. (11) for repeated eigenvalue derivatives are solved. This completes the computation of Z'.

Equation (15) also gives the off-diagonal elements of E and the third derivatives of the eigenvalues. If the second derivatives of the eigenvectors Z'' are also desired, one needs only to compute the remaining diagonal elements of E. This is done by twice differentiating the equation $Z^TMZ = I$ and substituting $Z'' = U + ZE$ yielding

$$E^T + E = -U^TMZ - Z^TMU - Z^TM''Z - 2(Z')^TM'Z - 2(Z')^TMZ' - 2Z^TM'Z' \equiv P \tag{17}$$

from which $e_{ii} = 0.5p_{ii}$. Note that, since e_{ij} and e_{ji} are indeterminate when $\lambda_j' = \lambda_i'$ in Eq. (16), the second derivatives of the eigenvectors with repeated eigenvalue derivatives are not complete.

IV. Example

Consider a simple 2×2 eigenvalue problem where

$$K = \frac{1}{p^2+1}\begin{pmatrix} p + p^2/2 + p^4/2 & -p/2 + p^2 - p^3/2 \\ -p/2 + p^2 - p^3/2 & 1/2 + p^2/2 + p^3 \end{pmatrix}$$

and $M = I \in R^{2\times 2}$. At $p = 1$, $K = I$ giving repeated eigenvalues $\lambda_1 = \lambda_2 = 1$, Choose $X = I$ as the corresponding eigenvectors. Information of derivatives of K and M with respect to p, evaluated at $p = 1$ is needed in the following computation and is given below: $M' = M'' = M''' = 0$, $K' = I$, and

$$K'' = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad K''' = \frac{1}{2}\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$$

The goal is to compute the eigenvector derivatives with repeated eigenvalues. Equation (6) is therefore used to find the matrix A. In this example, $D = I(I - 0)I = I$ yields the repeated eigenvalues $\lambda_1' = \lambda_2' = 1$. The corresponding eigenvectors A are indeterminate and are chosen to be the identity matrix I. Since Eq. (6) fails to give a unique transformation matrix A, Eq. (13) is required to solve for B and Λ''. D_1 is computed to be equal to K'', which gives $\lambda_1'' = 0$, $\lambda_2'' = 1$, and the corresponding unique eigenvectors

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Therefore, the correct eigenvectors $Z = XAB = B$ are obtained. In the computation of D_1, $\bar{G} = I$ and $\bar{F}_1 = 0$ have been used.

To obtain Z', $V = 0$ is obtained from Eq. (8), and Eq. (9) gives $Q = 0$, from which $c_{11} = c_{22} = 0$ and $c_{12} = -c_{21} = c$, where c is an undetermined constant. The resulting eigenvectors $Z' = V + ZC$ are then

$$Z' = \frac{1}{\sqrt{2}}\begin{pmatrix} c & c \\ -c & c \end{pmatrix}$$

U in Z'' vanishes since $\bar{H} = 0$. S in Eq. (15) is computed as $s_{11} = s_{22} = 0$, $s_{12} = s_{21} = -\tfrac{1}{2} - c$. Hence, it follows from $s_{12} = s_{21} = 0$ that $c = -\tfrac{1}{2}$. The resulting Z' is now determined as

$$Z' = \frac{1}{2\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

The eigenvalues and the eigenvectors as functions of the parameter p of the example problem are given as $\lambda_1 = p$, $x_1 = (1, p)^T/\sqrt{1+p^2}$, $\lambda_2 = (1 + p^2)/s$, and $x_2 = (-p, 1)^T/\sqrt{1+p^2}$. Numerical results given earlier match these exactly. Other choices of eigenvectors different from the correct ones chosen here, when eigenvalue derivatives are repeated, will generally lead to incorrect results. For example, if one uses $Z = XA = I$ and follow Dailey's algorithm in Ref. 5, one obtains incorrect results: $\lambda_1'' = \lambda_2'' = \tfrac{1}{2}$ and $Z' = 0$.

V. Conclusions

A new method based on an extension of the previously published algorithms is presented to handle the modal sensitivity problem when repeated eigenvalues and repeated eigenvalue derivatives are present. This new algorithm preserves the merits of the previous ones and produces accurate results. More information on the next-level derivatives (i.e., the third derivatives of eigenvalues and the second derivatives of eigenvectors associated with distinct eigenvalue derivatives) are also obtained, as opposed to the previous methods.

Acknowledgments

This material is based in part on work supported by Texas Advanced Research Program under Grant 999903-069.

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